Inverse trig relations

Suppose you have a value y of the sine function or the cosine function, and you want to find the inverse of that function at y (i.e., you want to find the angle θ such that $\sin\theta = y$ or $\cos\theta = y$). Well, in order for any function $f(x)$ to have an inverse, that function must satisfy the condition that for every element y of the range of f, there is only one element x of the domain of f such that $f(x) = y$. Since the sine and cosine functions are periodic (i.e., each value of the sine function is repeated over and over, and the same is true of every value of the cosine function), that condition isn't satisfied.

For every element y of the range of the basic sine function ($\sin\theta$), there are infinitely many angles θ such that $\sin\theta = y$. Similarly, for every element y of the range of the basic cosine function ($\cos\theta$), there are infinitely many angles θ such that $\cos\theta = y$.

Thus we'll use the term inverse sine relation (instead of the term inverse sine function) in referring to the whole set of angles θ such that $\sin\theta$ is equal to a given element of the range of the sine function. Similarly, we'll use the term inverse cosine relation (instead of inverse cosine function) in referring to the whole set of angles θ such that $\cos\theta$ is equal to a given element of the range of the cosine function.

Recall that for every angle θ and every integer n,

$$\sin(\theta + 2n\pi) = \sin\theta$$

$$\cos(\theta + 2n\pi) = \cos\theta$$

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The range of both the basic sine function and the basic cosine function is the interval $[-1,1]$. Thus if y is an element of $[-1,1]$ and θ is some angle such that $\sin\theta = y$, then the inverse sine relation for that value of y contains (at least) all the angles α for which there is some integer n such that

$$\alpha = \theta + 2n\pi$$

Similarly, if y is an element of $[-1,1]$ and θ is some angle such that $\cos\theta = y$, then the inverse cosine relation for that value of y contains (at least) all the angles α for which there is some integer n such that

$$\alpha = \theta + 2n\pi$$

Now let y be any element of the interval $[-1,1]$. If we fix a particular angle θ such that $\sin\theta = y$, do we have the entire inverse sine relation for that value of y if we include only the angles α such that $\alpha = \theta + 2n\pi$ for some integer n? Also, if we fix a particular angle θ such that $\cos\theta = y$, do we have the entire inverse cosine relation for that value of y if we include only the angles α such that $\alpha = \theta + 2n\pi$ for some integer n?

For all but just two values of y (namely, 1 and -1), the answer to both questions is No. To see this, we'll apply some trigonometric identities.

By the difference identity for sine,

$$\sin(\pi - \theta) = \sin\pi\cos\theta - \cos\pi\sin\theta = (0)\cos\theta - (-1)\sin\theta = \sin\theta$$

The difference between θ and $\pi - \theta$ is

$$\theta - (\pi - \theta) = \theta - \pi + \theta = 2\theta - \pi$$

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This difference $(2\theta - \pi)$ is a multiple of 2π if and only if there is some integer n such that

$$2\theta - \pi = 2n\pi$$

Now

$$2\theta - \pi = 2n\pi \iff 2\theta = (2n+1)\pi \iff \theta = \frac{(2n+1)\pi}{2}$$

An integer can be written in the form $2n+1$ if and only if that integer is odd, so the indicated difference $(2\theta - \pi)$ is a multiple of 2π if and only if θ is an odd multiple of $\pi/2$, which occurs if and only if $\sin\theta = \pm 1$.

By the even identity for cosine,

$$\cos(-\theta) = \cos\theta$$

The difference between θ and $-\theta$ is

$$\theta - (-\theta) = \theta + \theta = 2\theta$$

This difference (2θ) is a multiple of 2π if and only if there is some integer n such that

$$2\theta = 2n\pi$$

Now

$$2\theta = 2n\pi \iff \theta = n\pi$$

Thus the indicated difference (2θ) is a multiple of 2π if and only if θ is a multiple of π. This occurs if and only if $\cos\theta = \pm 1$.

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What we have found is that if $y \in [-1,1]$ and θ is some angle such that $\sin\theta = y$, then the inverse sine relation for that value of y includes (at least) all the angles α for which there is some integer n such that

$$\alpha = \theta + 2n\pi \qquad \text{or} \qquad \alpha = \pi - \theta + 2n\pi$$

Similarly, we have found that if θ is some angle such that $\cos\theta = y$, then the inverse cosine relation for that value of y includes (at least) all the angles α for which there is some integer n such that

$$\alpha = \theta + 2n\pi \qquad \text{or} \qquad \alpha = -\theta + 2n\pi$$

As it turns out, these more-inclusive sets of angles do, in fact, constitute the entire inverse sine relation and the entire inverse cosine relation for that value of y. To show this, it suffices to verify that they include all the angles in the interval $[0,2\pi)$ that have the stated properties, because if α is an angle which is not in the interval $[0,2\pi)$, then there exist an angle θ in the interval $[0,2\pi)$ and an integer n such that $\sin\alpha = \sin\theta + 2n\pi$.

There is just one angle α in the interval $[0,2\pi)$ whose sine is equal to 1 (namely, $\pi/2$), and there is just one angle in the interval $[0,2\pi)$ whose sine is equal to -1 (namely, $3\pi/2$).

There are two angles in the interval $[0,2\pi)$ whose sine is equal to 0: 0 and π. If we let $\alpha = \pi$ and $\theta = 0$, then

$$\alpha = \pi = \pi - 0 = \pi - 0 + 2(0)\pi = \pi - \theta + 2(0)\pi$$

What remains to be verified for sine is the case where y is in $(-1,0) \cup (0,1)$. If $y \in (0,1)$, there are two angles in the interval $[0,2\pi)$ whose sine is equal to y: one in the first quadrant and one in the second quadrant. If we let α be the

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angle in the second quadrant whose sine is equal to y, and θ the angle in the first quadrant whose sine is equal to y, then θ is the reference angle for both α and θ, so

$$\alpha = \pi - \theta = \pi - \theta + 2(0)\pi$$

If $y \in (-1,0)$, there are two angles in the interval $[0,2\pi)$ whose sine is equal to y: one in the third quadrant and one in the fourth quadrant. If we let α be the angle in the third quadrant whose sine is equal to y, and θ the angle in the fourth quadrant whose sine is equal to y, then there is an acute angle β such that β is the reference angle for both α and θ. Therefore,

$$\alpha = \pi + \beta$$

and

$$\theta = 2\pi - \beta$$

Solving the latter equation for β, we obtain

$$\beta = 2\pi - \theta$$

Substituting this value of β into the equation for α, we get

$$\alpha = \pi + \beta = \pi + (2\pi - \theta) = \pi - \theta + 2\pi = \pi - \theta + 2(1)\pi$$

There is just one angle α in the interval $[0,2\pi)$ whose cosine is equal to 1 (namely, 0), and there is just one angle in the interval $[0,2\pi)$ whose cosine is equal to -1 (namely, π).

There are two angles in the interval $[0,2\pi)$ whose cosine is equal to 0: $\pi/2$ and $3\pi/2$. If we let $\alpha = 3\pi/2$ and $\theta = \pi/2$, then

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$$\alpha = \frac{3\pi}{2} = 2\pi - \frac{\pi}{2} = -\frac{\pi}{2} + 2\pi = -\frac{\pi}{2} + 2(1)\pi = -\theta + 2(1)\pi$$

What remains to be verified for cosine is the case where y is in $(-1,0) \cup (0,1)$. If $y \in (0,1)$, there are two angles in the interval $[0,2\pi)$ whose cosine is equal to y: one in the first quadrant and one in the fourth quadrant. If we let α be the angle in the fourth quadrant whose cosine is equal to y, and θ the angle in the first quadrant whose cosine is equal to y, then θ is the reference angle for both α and θ, so

$$\alpha = 2\pi - \theta = -\theta + 2\pi = -\theta + 2(1)\pi$$

If $y \in (-1,0)$, there are two angles in the interval $[0,2\pi)$ whose cosine is equal to y: one in the second quadrant and one in the third quadrant. If we let α be the angle in the second quadrant whose cosine is equal to y, and θ the angle in the third quadrant whose cosine is equal to y, then there is an acute angle β such that β is the reference angle for both α and θ. Therefore,

$$\alpha = \pi - \beta$$

and

$$\theta = \pi + \beta$$

Solving this latter equation for β, we obtain

$$\beta = \theta - \pi$$

Substituting this value of β into the equation for α, we get

$$\alpha = \pi - \beta = \pi - (\theta - \pi) = \pi - \theta + \pi = -\theta + 2\pi = -\theta + 2(1)\pi$$

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Thus we have established that if $y \in [-1,1]$ and θ is any angle whose sine is equal to y, then an angle α is in the inverse sine relation for that value of y if and only if there is some integer n such that

$$\alpha = \theta + 2n\pi \qquad \text{or} \qquad \alpha = \pi - \theta + 2n\pi$$

Now $\pi = 180°$, so in degrees this would be

$$\alpha = \theta + 2n(180°) \qquad \text{or} \qquad \alpha = 180° - \theta + 2n(180°)$$

which can be written as

$$\alpha = \theta + n(360°) \qquad \text{or} \qquad \alpha = 180° - \theta + n(360°)$$

Example

Find all angles whose sine is equal to $1/2$. Express this in both radians and degrees.

We know that $\sin(\pi/6) = 1/2$, so in radians the indicated set is the set of all angles α for which there is some integer n such that

$$\alpha = \frac{\pi}{6} + 2n\pi \qquad \text{or} \qquad \alpha = \pi - \frac{\pi}{6} + 2n\pi$$

Now

$$\pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

so we could write that as

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$$\alpha = \frac{\pi}{6} + 2n\pi \qquad \text{or} \qquad \alpha = \frac{5\pi}{6} + 2n\pi$$

We also know that $\pi/6 = 30°$, so $5\pi/6 = 5(30°) = 150°$, hence in degrees this would be

$$\alpha = 30° + n(360°) \qquad \text{or} \qquad \alpha = 150° + n(360°)$$

Also, we have established that if $y \in [-1,1]$ and θ is any angle whose cosine is equal to y, then an angle α is in the inverse cosine relation for that value of y if and only if there is some integer n such that

$$\alpha = \theta + 2n\pi \qquad \text{or} \qquad \alpha = -\theta + 2n\pi$$

In degrees:

$$\alpha = \theta + n(360°) \qquad \text{or} \qquad \alpha = -\theta + n(360°)$$

Example

Find all angles whose cosine is equal to -1. Express this in both radians and degrees.

We know that $\cos \pi = -1$, so in radians the indicated set is the set of all angles α for which there is some integer n such that

$$\alpha = \pi + 2n\pi \qquad \text{or} \qquad \alpha = -\pi + 2n\pi$$

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In this case, however, we don't need the clause after the word "or," because the only angle in the interval $[0,2\pi)$ whose cosine is equal to -1 is π. (You might want to convince yourself that an angle which can be written as $-\pi + 2n\pi$ for some integer n can also be written as $\pi + 2n\pi$ for a different integer n.) Thus the indicated set reduces to the set of all angles α for which there is some integer n such that

$$\alpha = \pi + 2n\pi$$

In degrees, this would be

$$\alpha = 180^\circ + n(360^\circ)$$

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